Lemma: If P, Q are any two partions on $[a,b]$, then

$$L(f,P) \leq U(f,Q).$$

proof: Consider a "common" refinement R of P and Q.

For example, $R = P \cup Q$. We know from last class that

$$\begin{aligned} L(f,P) &\le L(f,R) \\ &\le U(f,R) \\ &\le U(f,Q). \qquad \square \end{aligned}$$

Theorem: (criterion for integrability)

Let f be bounded, $f:[a,b]\to\mathbb{R}$.

Then f is integrable on $[a,b]$ if and only if $\forall\ \varepsilon>0$, $\exists$ partition P of $[a,b]$ such that

$$\boxed{U(f,P)-L(f,P)<\varepsilon}$$

proof: $\Rightarrow$ Suppose f is integrable.

Then

$$U(f) = \inf\{U(f,P) \mid P \text{ partions } [a,b]\}$$

$$= \sup\{L(f,P) \mid P \text{ partitions } [a,b]\}$$

$$= L(f).$$

Then $\exists$ a partition P of $[a,b]$ such that $U(f,P) - U(f) < \frac{\varepsilon}{2}$ and a partition Q of $[a,b]$ with $L(f) - L(f,Q) < \frac{\varepsilon}{2}$.

Let R be a common refinement of P and Q.

Then

$$L(f,Q) \leq L(f,R)$$
$$\leq U(f,R) \leq U(f,P).$$

Then

$$\varepsilon/2 > L(f) - L(f,Q)$$
$$> L(f) - L(f,R) \quad \text{and}$$

$$\varepsilon/2 > U(f,P) - U(f)$$
$$> U(f,R) - U(f)$$

Then

$$U(f,R)-L(f,R)$$

$$= U(f,R) - U(f) + U(f) - L(f,R)$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\parallel$ integrability

$$= U(f,R) - U(f) + L(f) - L(f,R)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

$\Leftarrow$ Suppose $\forall\ \varepsilon>0$, $\exists$ partition P with

$$U(f,P)-L(f,P)<\varepsilon$$

Then $U(f) \leq U(f,P)$

and $L(f) \geq L(f,P)$, so

$$U(f) - L(f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

we have $U(f) = L(f)$. $\square$

Theorem: (continuity) Let f be continuous on $[a,b]$. Then f is integrable on $[a,b]$.

proof: The interval $[a,b]$ is compact, so f is uniformly continuous on $[a,b]$. Therefore, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon/_{b-a} \quad \text{whenever}$$

$x, y \in [a,b]$ and $|x-y| < \delta$.

There exists an $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$.

Let P be the partition given by

$$x_i = a + \frac{i(b-a)}{n}.$$

Then $$U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

But $x_i - x_{i-1} = \frac{b-a}{n} \ \forall 1 \le i \le n,$

so $$U(f,P) = \frac{b-a}{n} \sum_{i=1}^{n} M_i$$

$$L(f,P) = \frac{b-a}{n} \sum_{i=1}^{n} m_i \ldots$$

$U(f,P) - L(f,P)$

$$= \frac{b-a}{n} \sum_{i=1}^{n} (M_i - m_i)$$

Since f is continuous on

$[x_{i-1}, x_i]$ $\forall\ 1 \le i \le n$,

f attains its maximum and

minimum on $[x_{i-1}, x_i]$, so

$\exists\ z_i, w_i \in [x_{i-1}, x_i]$ $\forall\ 1 \le i \le n$,

$$M_i = f(z_i),\ m_i = f(w_i).$$

But $|z_i - w_i| \le x_i - x_{i-1}$

$$= \frac{b-a}{n}$$

$$< \delta.$$

This implies that

$$M_i - m_i$$
$$= |f(z_i) - f(w_i)|$$
$$< \frac{\varepsilon}{b-a}, \quad \text{so}$$

$$U(f,P) - L(f,P) = \frac{b-a}{n} \sum_{i=1}^{n} (M_i - m_i)$$
$$< \frac{\cancel{b-a}}{n} \cdot \frac{\varepsilon}{\cancel{b-a}} \cdot \sum_{i=1}^{n} 1$$
$$= \frac{\varepsilon}{n} \cdot n = \varepsilon.$$

So by the previous lemma, we have shown that f is integrable. $\square$

What about functions with finitely many discontinuities on a closed interval?

We know this is not an issue, yet we still require a proof! It will come in two steps.

Theorem: (discontinuity at an endpoint) Suppose f is bounded on $[a,b]$ and $\forall\, c,\ a < c \le b$, f is integrable on $[c,b]$. Then f is integrable on $[a,b]$.

proof: Since f is bounded on $[a,b]$, $\sup\limits_{x\in[a,b]} f(x) = M$ and

$\inf\limits_{x\in[a,b]} f(x) = m$ exist.

Let $\varepsilon > 0$.

Let $N = \max\{|M|, |m|\}$

Choose $P > N$ so that

$$c = a + \frac{\varepsilon}{4P} < b.$$

Then $\exists$ a partion Q of $[c,b]$ with

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}.$$

Then $R = \{a\} \cup Q$ is a partition of $[a,b]$.

$$U(f,R) - L(f,R)$$

$$= (M_0 - m_0)\frac{\varepsilon}{4P} + U(f,Q) - L(f,Q)$$

$$< (M_0 - m_0)\frac{\varepsilon}{4P} + \frac{\varepsilon}{2}.$$

But $M_0 - m_0 < M - m < 2N$
$< 2P$, so

$$U(f,R) - L(f,R)$$
$$< 2P\left(\frac{\varepsilon}{4P}\right) + \frac{\varepsilon}{2} < \varepsilon,$$

by lemma, we are done. $\square$

$M =$ bound.

disc at $a = x$.

Consider the interval

$$\left[a, a + \frac{\varepsilon}{4M}\right].$$

Then f is integrable

on $\left[a + \frac{\varepsilon}{4M}, b\right]$,

So let P be any

partition of $\left[a + \frac{\varepsilon}{2M}, b\right]$

s.t.

$$U(f,P) - L(f,P) < \frac{\varepsilon}{2}$$

Then if Q is the partition

$Q = \{a, P\}$ of $[a,b]$,

$$L(f,Q) = m_0 \frac{\varepsilon}{4M} + L(f,P)$$

$$U(f,Q) = M_0 \frac{\varepsilon}{4M} + U(f,P)$$

$$\begin{aligned} U(f,Q) - L(f,Q) &= U(f,P) - L(f,P) \\ &\quad + (M_0 - m_0)\frac{\varepsilon}{4M} \\ &< \frac{\varepsilon}{2} + \frac{\cancel{2}\cancel{M}\varepsilon}{2\cancel{4}\cancel{M}} \\ &= \varepsilon \end{aligned}$$